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Nonintegrability and chaos in linearly and nonlinearly coupled quartic oscillators : Melnikov analysis

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Abstract Using the Holmes-Marsden approach of Melnikov analysis nonintegrability and chaotic behaviour of two coupled quartic oscillators are analytically studied Coupling is linear in one case and nonlinear in the other case Both systems are proved to be nonintegrable and chaotic Width of chaotic layer is also estimated

Keywords Chaos and nonintegrability, Melnikov analysis, anharmonic oscillators

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1. Introduction

During the last two decades, the existence and importance of chaos in Hamiltonian systems have been well established. Although heavily studied numerically, there have been very few attempts to rigorously establish chaos in these systems. One of the characteristic features of chaos is the sensitive dependence of the solution on the initial conditions. In contrast to chaotic, nonintegrable systems there also exist integrable nonlinear dynamical systems which show regular, predictable and ordered behaviour. But they are rather exceptional and there are no general techniques to identify them. In Hamiltonian systems, the question of existence of chaos is intimately related to whether the system is integrable or not. Though, Painlevé analysis helps one in identifying whether the system is integrable or not, it is still based on a conjecture. However, most of the studies on chaotic behaviour of deterministic nonlinear dynamical systems so far done have been numerical employing techniques such as Poincaré surface of sections, calculation of Lyapunov exponents, etc. Because of the limitations of the accuracy of numerical methods, establishment of chaos in many systems is difficult, if not impossible. Though such

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techniques help us to understand much about the system, rigorous analytical results establish the fact unambiguously.

In this context, we would like to establish the existence of chaos rigorously in certain anharmonic oscillators using Melnikov method. The technique developed by Melnikov [1] is an analytical method to detect chaos in a system. It was originally developed for dissipative systems. Holmes and Marsden [2-4] extended the method to Hamiltonian systems. The essence of this method is to check whether homoclinic manifolds intersect transversally in which case the existence of Smale horseshoes and hence of chaos will follow. In the case of Hamiltonian systems existence of horseshoes always implies asymptotic chaos characterised by positive maximal Lyapunov exponent, unlike dissipative systems where there are transients [5]. The technique is also useful in estimating the width of the chaotic layer. This technique was used in a few number of earlier works, for example, in the study of chaotic orbits of galactic potentials [6], in studying stability of Maxwell-Bloch equations [7], in the context of controlling chaos in Hamiltonian systems [8], *etc*. The effectiveness of this analytical method in studying Hamiltonian chaos can be fruitfully exploited in other systems also.

In this paper, we study the nonintegrability and chaotic behaviour of two coupled quartic oscillators (i) with nonlinear coupling and (ii) with linear coupling, using the modified Melnikov method. The oscillators without the coupling term is trivially integrable. Here we investigate the effect of linear coupling of two anharmonic oscillators in comparison with the nonlinear coupling. Both systems are shown to be nonintegrable and chaotic. The width of the chaotic layer is also estimated. In several physical situations, quartic potentials represent the first correction beyond the harmonic approximation. They are used as models in condensed matter physics, field theory, astrophysics, optics, molecular dynamics, *etc*. The system of N coupled quartic oscillators known as Fermi-Pasta-Ulam chain has been studied as a paradigm for microscopic route to thermodynamics.

The rest of the paper is organised as follows. In the next section, Melnikov method as applied to a Hamiltonian system with two degrees of freedom is briefly explained. In Section 3, details regarding the systems under study and the Melnikov integral calculations are given. Section 4 summarises our findings. Some of the details of the calculations are given in the appendices.

2. Melnikov-Holmes-Marsden method

Holmes and Marsden extended the method of Melnikov to study small perturbations of integrable Hamiltonians [1-3]. Consider a system with two degrees of freedom having Hamiltonian of the form,

$$H^{\varepsilon}(q,p,\theta,I) = F(q,p) + G(I) + \varepsilon H^{1}(q,p,\theta,I), \tag{1}$$

where q is a generalised coordinate, p its conjugate momentum and I, θ are action-angle variables. When $\varepsilon = 0$, the unperturbed system is completely integrable. We assume that $\Omega(I) = \partial G/\partial I > 0$ for $I > 0$ and the (q, p) phase plane contains a hyperbolic saddle point (q_0, P_0) with a homoclinic orbit $(\bar{q}(t), \bar{p}(t))$ *ie*., (q_0, P_0) lies in a closed curve of F.

Let $h = H^{\varepsilon}$ be the total energy of the system. Pick $h > h'$ and let $I^0 = G^{-1}(h - h')$, where $h' = F(\bar{q}, \bar{p})$ be the energy of the homoclinic manifold of the F system. Let $\{F, H^1\}(t - t_0)$ be the Poisson bracket of F and H^1 evaluated along an orbit $(\bar{q}(t-t_0), \bar{p}(t-t_0), \Omega(I^0)t, I^0)$ in the homoclinic manifold. Holmes and Marsden [2] proved that if the Melnikov function

$$M(h,t_0)=\int_{-\infty}^{\infty}\{F,H^1\}(t-t_0)\,dt \tag{2}$$

has simple zeros, the Hamiltonian system (1) has a Smale horseshoe in its dynamics on the energy surface $H^{\varepsilon}=h$ hence it is nonintegrable and chaotic.

The width of chaotic layer is given by

$$d(t_0)=\varepsilon\int_{-\infty}^{\infty}\frac{\{F,H^1\}(t-t_0)\Omega(I^0)dt}{\Omega^2(I^0)}\Bigg/\frac{\left|\left(\frac{\partial F}{\partial p}(\bar{q}(0),\bar{p}(0)),\frac{\partial F}{\partial q}(\bar{q}(0),\bar{p}(0))\right)\right|}{\Omega(I^0)}+O(\varepsilon^2)$$

$$=\varepsilon M(t_0)\Bigg/\left|\left(\frac{\partial F}{\partial p}(\bar{q}(0),\bar{p}(0)),\frac{\partial F}{\partial q}(\bar{q}(0),\bar{p}(0))\right)\right|+O(\varepsilon^2). \tag{3}$$

Maximum width of splitting of the manifolds and hence of the chaotic layer near the closed curve $(\bar{q}(0), \bar{p}(0), \Omega t_0, I^0)$ on the energy surface $H^{\varepsilon}=h$ is given by

$$d_{max}=\varepsilon\sup_{t_0\in R}M(h,t_0)\Bigg/\left|\left(\frac{\partial F}{\partial p}(\bar{q}(0),\bar{p}(0)),\frac{\partial F}{\partial q}(\bar{q}(0),\bar{p}(0))\right)\right|+O(\varepsilon^2). \tag{4}$$

In [4] an application of this method to prove the nonintegrability of the Hènon-Heiles Hamiltonian is given. It is to be noted that the unperturbed G system need not necessarily be transformed to action angle variables, all that requires is that it has a continuous family of periodic orbits in some (compact) subspace of its phase space. The method can be easily extended to the case in which the unperturbed system, while integrable does not decouple into two independent systems.

3. The quartic oscillators and chaos

We study Hamiltonians of the form,

$$H^{\varepsilon}=F_1(x,p_x)+F_2(y,p_y)+\varepsilon H^1, \tag{5}$$

where
$$F_1(x,p_x)=p_x^2/2-x^2/2+ax^4, \tag{6}$$

$$F_2(y,p_y)=P_y^2/2-y^2/2+by^4. \tag{7}$$

We calculate Melnikov integral in two cases of the coupled oscillators :

(a) the case of nonlinear coupling :

$$H^1=x^2y^2, \tag{8}$$

and (b) the case of linear coupling :

$$H^1=xy. \tag{9}$$

$$= -\int_{-\infty}^{\infty} p_x . 2xy^2 (t - t_0) dt . \tag{25}$$

Substituting for x, p_x, y, we get

$$M(h, t_0) = -\int_{-\infty}^{\infty} \frac{e^2}{a\sqrt{2b}} \operatorname{sech}^2 (t - t_0) \tanh (t - t_0) \, dn^2 [e(2b)^{1/4} t, k] dt$$

$$= -\int_{-\infty}^{\infty} \frac{e^2}{a\sqrt{2b}} \operatorname{sech}^2 (t - t_0) \tanh (t - t_0) \{1 - k^2 sn^2 [e(2b)^{1/4} t, k]\} dt . \tag{26}$$

RHS of (26) can be considered as the sum of two terms The first integral vanishes because of the fact that the integrand is an odd function. The integral is evaluated to obtain (details are given in the Appendix A),

$$M(h, t_0) = \frac{ek^2}{(8a^2b)^{1/2}(2b)^{1/4}} \sum_{m=0}^{\infty} C_m^2 \pi \frac{v_m^2}{2} \operatorname{csch} \frac{\pi v_m}{2} \sin 2\mu_m t_0 , \tag{27}$$

where $$v_m = 2\mu_m = \frac{(2m+1)\pi e(2b)^{1/4}}{2K(k)} \tag{28}$$

and K is the complete elliptic integral of the first kind. Equation (27) shows that $M(h, t_0)$ has simple zeros at $t_0 = 0$. That is $M(h, t_0) = 0$ when $t_0 = 0$ and $\frac{\partial M}{\partial t_0}(h, t_0) \neq 0$. Thus we find that the stable and unstable manifolds of the system intersect transversally and hence the system is chaotic and does not possess a second integral of motion.

Width of the chaotic layer

Using $x(0) = 1/\sqrt{2a}$, $p_x(0) = 0$, from (4), we obtain the maximum width as

$$d_{max} = \varepsilon \sup_{t_0 \varepsilon R} M(h, t_0) / \frac{1}{\sqrt{2a}} . \tag{29}$$

The direct evaluation of the numerator for all energy values is very difficult. However, we can find its value in a special case when the energy of the F_2 system orbit is the minimum. Then $E_0 = -1/16b$ and $g^2 = 1/2\sqrt{2b}$, $e^2 = 1/2\sqrt{2b}$, $k^2 = 0$. In this approximation

$$M(h, t_0) = -\frac{e^2}{a\sqrt{2b}} \int_{-\infty}^{\infty} \operatorname{sech}^2 (t - t_0) \tanh (t - t_0) \{1 - k^2 \sin^2 [e(2b)^{1/4} t]\} dt, \tag{30}$$

because $sn^2(u, k) = \sin^2 u$ when $k \approx 0$. With a transformation $t = t + t_0$, we obtain

$$M(h, t_0) = \frac{e^2k^2}{(8a^2b)^{1/2}} \int_{-\infty}^{\infty} \operatorname{sech}^2 (t) \tanh (t) [1 - \cos\sqrt{2}(t + t_0)] dt$$

$$- \frac{e^2k^2}{(8a^2b)^{1/2}} \sin\sqrt{2}t_0 \int_{-\infty}^{\infty} \operatorname{sech}^2 (t) \tanh (t) \sin\sqrt{2} t dt . \tag{31}$$

It can be evaluated using the method of residues [21]

$$M(h, t_0) = \frac{e^2 k^2}{(8a^2 b)^{1/2}} \sin\sqrt{2} t_0 \pi \operatorname{csch} \frac{\pi}{\sqrt{2}}, \tag{32}$$

$$d_{max} = \varepsilon \frac{e^2 k^2}{(4ab)^{1/2}} \operatorname{csch} \frac{\pi}{\sqrt{2}}. \tag{33}$$

3.2 $H' = xy$, the case of linear coupling :

Here,

$$M(h, t_0) = -\int_{-\infty}^{\infty} p_x (t - t_0)\, y(t) dt. \tag{34}$$

Substituting for y, p_x from (17) and (24) and putting $t = t + t_0$ and $e(2b)^{1/4} (t + t_0) = s$ we get

$$M(h, t_0) = -\frac{1}{2\sqrt{ab}} \int_{-\infty}^{\infty} \operatorname{sech}(t) \tanh(t) [1 - k^2 sn^2 (s, k)]^{1/2} ds. \tag{35}$$

Expanding $[1 - k^2 sn^2 (s, k)]^{1/2}$, we get

$$M(h, t_0) = -\frac{1}{2\sqrt{ab}} \int_{-\infty}^{\infty} \operatorname{sech}(t) \tanh(t) \{1 - k^2 sn^2 [s, k] +$$

$$k^4 sn^4 (s, k) - \ldots\ldots\} ds$$

$$= I_0 + I_1 + I_2 + I_3 + \ldots\ldots\ldots \tag{36}$$

The first term of the integral is zero. Other terms contain even powers of sn. All of them can be shown to be possessing simple zeros (See Appendix B). Hence the linearly coupled quartic oscillator system is also nonintegrable and chaotic.

The width of the stochastic layer in this case can be estimated as in the case of nonlinear coupling when $k \approx 0$ so that k^4 and other higher order terms can be neglected.

4. Conclusion

In this paper, we have studied analytically the nonintegrability and chaotic behaviour of two cases of coupled quartic oscillators. Both of them are proved to be chaotic. In one case, the coupling is linear and in the other case, it is nonlinear. Without these terms the oscillator is integrable. We used the Melnikov method of Holmes and Marsden. Width of chaotic layer has also been estimated. The strength of perturbation is assumed to be very small ($\varepsilon << 1$). We must notice that the method is not applicable for any value of ε. For example when $b = a$, $\varepsilon = 2a$, and $\varepsilon = 6a$, $b = 1/8a$, $\varepsilon = 3/4a$, and $b = 1/16a$, $\varepsilon = 3/4a$, the system with nonlinear coupling is integrable. There are no known integrable cases in the system with linear coupling. Melnikov-Holmes-Marsden method is an effective tool in studying chaos in Hamiltonian systems and at present we are investigating chaos in N coupled oscillators (FPU chain) using this technique. It is interesting to note that, chaotic behaviour of some systems is analytically established and studied without recourse to numerical computation.

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Appendix A

To evaluate (26), we expand the elliptic sine *sn* in Fourier series [20] as

$$sn(s,k)=\frac{2\pi}{kK(k)}\sum_{m=0}^{\infty}\frac{q^{m+1/2}}{1-q^{2m+1}}\sin\left[(2m+1)\frac{\pi s}{2K(k)}\right], \tag{A1}$$

where $K(k)$ is the complete elliptic integral of the first kind, $K'(k)$ is the complementary elliptic integral and $q=\exp(-\pi K'(k)/K(k))$ is the elliptic norm. (A1) can be written as

$$sn(s,k)=\sum_{m=0}^{\infty}C_m\sin(\lambda_m s)\equiv\sum_{m=0}^{\infty}C_m\sin(\mu_m t), \tag{A2}$$

where
$$C_m=\frac{2\pi}{kK(k)}\frac{q^{m+1/2}}{1-q^{2m+1}} \tag{A3}$$

and
$$\mu_m = (2m+1)\frac{\pi e(2b)^{1/4}}{2K(k)} \tag{A4}$$

Then

$$M(h,t_0) = -\frac{ek^2}{a\sqrt{2b}}\frac{1}{(2b)^{1/4}}\int_{-\infty}^{\infty}\mathrm{sech}^2(t-t_0)\tanh(t-t_0)\sum_{m=0}^{\infty} C_m \sin\mu_t \quad dt\,. \tag{A5}$$

A transformation $t \to t+t_0$ gives

$$M(h,t_0) = -\frac{ek^2}{a\sqrt{2b}}\frac{1}{(2b)^{1/4}}\int_{-\infty}^{\infty}\mathrm{sech}^2 t\tanh t\sum_{m=0}^{\infty} C_m \sin\mu_m(t+t_0) \quad dt\,. \tag{A6}$$

Expanding the series in the bracket we get terms of the type

$$C_i C_j \sin\mu_i t \sin\mu_j t \cos\mu_i t_0 \cos\mu_j t_0\,,$$

$$C_i C_j \sin\mu_i t \cos\mu_j t \cos\mu_i t_0 \sin\mu_j t_0\,,$$

$$C_i C_j \cos\mu_i t \sin\mu_j t \sin\mu_i t_0 \cos\mu_j t_0\,,$$

$$C_i C_j \cos\mu_i t \cos\mu_j t \sin\mu_i t_0 \sin\mu_j t_0\,.$$

Terms of the first and fourth type vanish because they are even in t. Second and third survive only when $i=j$, due to orthogonality of Fourier components. Hence,

$$M(h,t_0) = -\frac{ek^2}{a\sqrt{2b}}\frac{1}{(2b)^{1/4}}\sum_{m=0}^{\infty} C_m^2 \sin 2\mu_m t_0\int_{-\infty}^{\infty}\mathrm{sech}^2 t\tanh t \sin 2\mu_m t dt$$

$$= \frac{ek^2}{(8a^2b)^{1/2}(2b)^{1/4}}\sum_{m=0}^{\infty} C_m^2\pi\frac{\nu_m^2}{2}\,\mathrm{csch}\,\pi\frac{\nu_m}{2}\sin 2\mu_m t_0\,. \tag{A7}$$

Appendix B

Let us consider the terms in (36). The first term I_0 is zero because it is odd.

$$I_1 = (Const.)\int_{-\infty}^{\infty}\mathrm{sech}\, t\tanh t\; sn^2(s,k)\, ds\,. \tag{B1}$$

Expanding sn in Fourier series and proceeding as in Appendix A, we obtain,

$$I_1 = (Const.)\int_{-\infty}^{\infty}\mathrm{sech}\, t\tanh t\sin 2\mu_m t dt\,. \tag{B2}$$

This integral can be evaluated using the result [21] as

$$\int_{-\infty}^{\infty} \sin \alpha x \frac{\sinh \beta x}{\cosh^2 \gamma x} dx = \frac{\pi\left[\alpha \sin \frac{\beta\pi}{2\gamma} \cosh \frac{\alpha\pi}{2\gamma} - \beta \cos \frac{\beta\pi}{2\gamma} \sinh \frac{\alpha\pi}{2\gamma}\right]}{\gamma^2\left(\cosh \frac{\alpha\pi}{\gamma} - \cos \frac{\beta\pi}{\gamma}\right)}, \tag{B3}$$

with $\beta < 2\gamma$. Therefore,

$$I_1 = (Const.) \sum_{m=0}^{\infty} 2\pi\mu_m \sin 2\mu_m t \operatorname{sech} \mu_m \pi .$$

I_1 has simple zeros.

$$I_2 = (Const.) \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \, sn^4 (s, k)\, ds . \tag{B4}$$

Expanding sn in Fourier series we have integrals with terms containing,

$$C_i C_j C_k C_l \sin \mu_i t \sin \mu_j t \sin \mu_k t \sin \mu_l t \cos \mu_i t_0 \cos \mu_j t_0 \cos \mu_k t_0 \cos \mu_l t_0 ,$$

$$C_i C_j C_k C_l \sin \mu_i t \sin \mu_j t \sin \mu_k t \cos \mu_l t \cos \mu_i t_0 \cos \mu_j t_0 \cos \mu_k t_0 \sin \mu_l t_0 ,$$

$$C_i C_j C_k C_l \sin \mu_i t \sin \mu_j t \cos \mu_k t \cos \mu_l t \cos \mu_i t_0 \cos \mu_j t_0 \sin \mu_k t_0 \sin \mu_l t_0 ,$$

$$C_i C_j C_k C_l \sin \mu_i t \cos \mu_j t \cos \mu_k t \cos \mu_l t \cos \mu_i t_0 \sin \mu_j t_0 \sin \mu_k t_0 \sin \mu_l t_0 ,$$

$$C_i C_j C_k C_l \cos \mu_i t \cos \mu_j t \cos \mu_k t \cos \mu_l t \sin \mu_i t_0 \sin \mu_j t_0 \sin \mu_k t_0 \sin \mu_l t_0 .$$

Contributions from first, third and fifth terms are zero because they are even and hence the corresponding integral is odd. Consider the second term. It can be written as

$$C_i C_j C_k C_l [\cos (\mu_i - \mu_j) t \sin(\mu_k + \mu_l) t + \cos (\mu_i - \mu_j) t \sin(\mu_k - \mu_l) t$$

$$- \cos (\mu_i + \mu_j) t \sin(\mu_k + \mu_l) t - \cos (\mu_i + \mu_j) t \sin(\mu_k - \mu_l) t]$$

$$\cos \mu_i t_0 \cos \mu_j t_0 \cos \mu_k t_0 \sin \mu_l t_0 .$$

Because of the othogonality of Fourier components the integral will be of the form

$$I_2 = (Const.) \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \sin 2pt ,$$

where p depends on μ. As in the case of I_1 this can be easily shown to possess simple zeros. For the higher order expansions I_3, I_4, ... , etc. also we get similar integrals by expanding the elliptic sine sn in terms of Fourier series and imposing the condition of orthogonality of Fourier components. Hence, the Melnikov function has simple zeros.